Optimal Measurement and Velocity Correction Programs for Midcourse Guidance

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The estimation and control of the perturbations in the preplanned midcourse portion of a space mission trajectory are considered The problem investigated is the optimization of the measurement program, the state estimator, and/or the feedback control gain program The performance index is a scalar function of the expected values of quadratic forms in the perturbations and/or in the errors in the estimates of perturbations Necessary conditions for the desired extremal solutions and a steepest-ascent computation procedure for obtaining approximate optimizations are presented A numerical example, in which the sequence of stars and near-body horizons to be sighted for the midcourse phase of a lunar mission is selected to minimize the terminal point uncertainty in position (or a weighted average of position and velocity uncertainties), is presented The numerical scheme is shown to converge to essentially the same result starting with various nominal measurement programs The minimum root-mean-square uncertainty in position at the terminal point is 10% less than the value obtained using the scheme suggested by R H Battin ["A statistical optimizing navigation procedure for space flight," ARS J 32, 1681–1692 (1962)]

1 Introduction

WE consider a space mission with a midcourse region which is nominally free-fall Conditions at booster burnout are predetermined so that a free fall from burnout will reach the terminal point with a specified velocity (The means of selecting this predetermined path will not be discussed here) Inevitable errors in the injection conditions require that some feedback control be used during midcourse There are three major elements in feedback control based on a reference path: 1) the measurement system which gathers position and/or velocity data; 2) the estimator which processes the data to determine a "best" estimate of the deviations from the preplanned trajectory; and 3) the controller which determines the corrective action to take Once the three elements are given, the deviations at the terminal point will depend only on the injection errors, the measurement errors, and the feedback control implementation errors We assume that these errors are random with known statistical distributions The effectiveness of the measurement and feedback gain programs may be evaluated by the statistical distribution of deviations (ensemble average quantities) at the terminal point The purpose of this paper is to present a means for achieving statistically optimal measurement and/or feedback gain programs

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2 Formulation of the Problem

The state of the system is represented by the six-vector‡ of position and velocity, denoted by x The nominal state history $x_{\mathrm{nom}}(t)$ is given (t is the independent variable time in this problem) Perturbations from the nominal state history resulting from the random errors and the feedback control corrections we denote as $\delta x(t)$ The estimate of the perturbations we denote as $\delta\hat{x}(t)$

The system is governed by a set of nonlinear ordinary differential equations

$$\dot{x} = f(x,u,v,t) \tag{2 1}$$

where $u(t)$ is the chosen control vector, three components of thrust; $v(t)$ is the vector of control implementation error; and f is a known six-vector of scalar functions of x,u,v,t The nominal path satisfies, in the midcourse region,

$$\dot{x}_{\mathrm{n\ m}} = f(x_{\mathrm{nom}},0,0,t) \tag{2 2}$$

which is merely the statement of free fall In this region, then, $u(t)$ is the feedback control and $v(t)$ the feedback control implementation error The nominal midcourse phase satisfies as many as six constraints:

$$\psi[x_{\mathrm{nom}}(t_f)] = 0 \tag{2 3}$$

Under the assumption of "small" perturbations in initial conditions, and in the desired control, the perturbation equations may be linearized to

$$\delta\dot{x} = F(t)\delta x + G(t)(u + v) \tag{2 4}$$

where $F(t) = (\partial f/\partial x)_{\mathrm{nom}}$, the ijth element of F is $(\partial f_i/\partial x_j)_{\mathrm{nom}}$, and $G(t) = (\partial f/\partial u)_{\mathrm{nom}}$ The measurements made are, in general, a p-vector set $M(x)$ The perturbation estimator, using the measurement data, we take as

$$\delta_{\dot{\hat{x}}} = F(t)\delta\hat{x} + G(t)u + K(t)[z - H(t)\delta\hat{x}] \tag{2 5}$$

where $K(t)$ is a $6 \times p$ gain matrix, to be selected, and

$$H(t) = (\partial M/\partial x)_{\mathrm{nom}} \tag{2 6}$$

Measurement errors $w(t)$ cannot be separated from the measurements The quantity z is the perturbation of the

‡ Vectors are column vectors unless otherwise noted

data received from its nominal value:

$$z = [M(x) + w] - M(x_{\text{nom}}) \qquad (2\ 7)$$

To the first order in δx,

$$z = H\delta x + w \qquad (2\ 8)$$

In this analysis we choose the control as linear feedback on the estimated state perturbation Thus,

$$u = -\Lambda\delta\hat{x} \qquad (2\ 9)$$

where $\Lambda(t)$ is a 3×6 gain matrix to be precalculated

It is easy to see from (2 4) and (2 5) that $\delta x(t)$ and $\delta\hat{x}(t)$ will be identically zero (to first order) if $\delta x(0) = \delta\hat{x}(0) = 0$ and $v(t) = w(t) = 0$ The initial (injection) deviation $\delta x(0)$ is, however, a random vector, and we wish to examine the ensemble average behavior of the midcourse region (in particular, the ensemble average terminal conditions)

Using (2 4) and (2 5) again, we conclude that the ensemble averages $\mathcal{E}[\delta x(t)]$ and $\mathcal{E}[\delta\hat{x}(t)]$ will be zero if $\mathcal{E}[\delta x(0)] = 0$ [$\delta\hat{x}(0)$ is assumed always zero] and if $v(t)$ and $w(t)$ have symmetric, zero-mean distributions Hence, we must employ at least second moments to gain any statistical measure of performance We shall measure statistical performance in terms of the quantities

$$\mathcal{E}(\psi^T A \psi)_{t=t_f} \qquad (2\ 10)$$

$$\mathcal{E}[(\hat{x} - x)^T B(\hat{x} - x)]_{t=t_f} \qquad (2\ 11)$$

and, where appropriate, the ensemble average of a measure of the control used Equation (2 10) measures the ensemble average quadratic deviation from the constraints (2 3) A is a symmetric matrix chosen, as desired, to weight the various components of ψ Equation (2 11) is the ensemble average of a quadratic form in the error in estimating the state B is also a chosen symmetric matrix The significance of (2 10) is obvious The uncertainty, as measured by (2 11), at the end of the midcourse phase exerts an important influence on the maneuvering efficiency in the next phase Note that (2 10) and (2 11) are ensemble-average quantities evaluated at the nominal terminal time Implicit in constraints (2 3) and performance measure (2 10) is the assumption of fixed terminal time The more general case of variable terminal time will be treated in Sec 6 §

3 Measurement Program That Minimizes Terminal Uncertainty

First, we consider the navigation problem by itself Here we define navigation to be the making of measurement(s) of function(s) of the state and the processing of the data obtained to make an estimate of the state To some extent, the choice of measurements may be open Further, having assumed an estimator of the form (2 5), we may select the gain matrix $K(t)$ We ask for the choice of measurements and the choice of $K(t)$ that will minimize the ensemble average terminal uncertainty as measured by

$$\phi = \mathcal{E}[(\hat{x} - x)^T B(\hat{x} - x)]_{t=t_f} \qquad (3\ 1)$$

As an obvious definition,

$$\hat{x}_{\text{n m}}(t) = x_{\text{n m}}(t) \qquad (3\ 2)$$

Thus, (3 1) reduces to

$$\phi = \mathcal{E}[(\delta\hat{x} - \delta x)^T B(\delta\hat{x} - \delta x)]_{t=t} \qquad (3\ 3)$$

Defining

$$E = \mathcal{E}(ee^T) \qquad (3\ 4)$$

§ The authors are indebted to Carlos Cavoti of General Electric Company, Philadelphia, for calling their attention to the question of variable terminal time, which was overlooked in the preprint of this paper

where

$$\begin{aligned} e &= \hat{x} - x \\ &= \delta\hat{x} - \delta x \end{aligned} \qquad (3\ 5)$$

is the error in the estimate, we write (3 3) as

$$\phi = \mathcal{E}(e^T B e)_{t=t} = \text{tr}BE_f \qquad (3\ 6)$$

where *tr* stands for trace and $(\ )_f$ is $(\ )_{t=t_f}$

To find conditions for minimizing (3 6), we use an approach learned from the doctoral research of Johansen[1] under A E Bryson Jr , at Harvard University First, we obtain the differential equation governing the covariance matrix E as

$$\begin{aligned} \dot{E} &= (d/dt)\mathcal{E}(ee^T) = \mathcal{E}(d/dt)(ee^T) \\ &= (F - KH)E + E(F - KH)^T + KRK^T + GQG^T \end{aligned} \qquad (3\ 7)$$

under the assumptions

$$\mathcal{E}[v(t)] = \mathcal{E}[w(t)] = 0$$
$$\mathcal{E}[v(t)v(\tau)^T] = Q(t)\delta(t - \tau)$$
$$\mathcal{E}[w(t)w(\tau)^T] = R(t)\delta(t - \tau)$$
$$\mathcal{E}[v(t)w(\tau)^T] = 0$$

where $Q(t)$ and $R(t)$ are known functions The convariance matrix $Q(t)$ may depend on the nominal path (which is given here), but in this section it may not depend upon the perturbation quantities This restriction will be partially relaxed in Sec 6 In addition, $R(t)$ there may depend upon the measurements

A particular (nominal) measurement sequence and esti mator gain matrix [with $E(0),Q(t),R(t)$] produce (nominal) ensemble-average results with covariance matrix $E(t)$ The first variation in $E(t)$ is governed by

$$\begin{aligned} \delta\dot{E} = (F - KH)\delta E + \delta E(F - KH)^T - \delta K\, HE - \\ K\delta HE - EH^T\delta K^T - E\delta H^T K^T + \\ \delta KR\, K^T + K\delta RK^T + KR\delta K^T + G\delta QG^T \end{aligned} \qquad (3\ 8)$$

We now consider, following Johansen, the matrix adjoint to δE If we choose

$$\dot{L}_E = -(F - KH)^T L_E - L_E(F - KH) \qquad (3\ 9)$$

where L_E is the (symmetric) adjoint to δE, we can multiply (3 8) on the left by L_E and (3 9) on the right by δE, add the results and take the trace to obtain

$$\begin{aligned} (d/dt)\ \text{tr}(L_E\delta E) = -2\ \text{tr}EL_EK\delta H - 2\text{tr}(L_EEH^T - \\ L_EKR)\delta K^T + \text{tr}K^TL_EK\delta R + \text{tr}G^TL_EG\delta Q \end{aligned} \qquad (3\ 10)$$

If we integrate (3 10) from $t = 0$ to $t = t_f$, we obtain

$$\text{tr}(L_E\delta E)_{t=t_f} = \text{tr}(L_E\delta E)_{t=0} + \int_0^{t_f} \left\{ \begin{matrix} -2\ \text{tr}EL_EK\delta H - 2\ \text{tr}(L_EEH - L_EKR)\delta K^T \\ + \text{tr}K^TL_EK\delta R + \text{tr}G^TL_EG\delta Q \end{matrix} \right\} dt \qquad (3\ 11)$$

With the choice

$$L_E(t_f) = B \qquad (3\ 12)$$

the left-hand side of (3 11) becomes $\delta\phi$, the variation in ϕ due to variations $\delta E(0)$, $\delta H(t)$, $\delta K(t)$, $\delta R(t)$, and $\delta Q(t)$

The first thing to notice in (3 11) is that, no matter what B is, and hence what $L_E(t)$ is, ϕ will be stationary with respect to a small variation $\delta K(t)$ if

$$K = EH^TR^{-1} \qquad (3\ 13)$$

By inspection of (3 11) we find the well known expression for the optimal gain for the navigation problem with quadratic performance index

It may be pointed out¶ that ϕ will be stationary with respect to $\delta K(t)$ for any matrix $EH^T - KR$ which is anni-

¶ This was done for the authors by an AIAA ıeviewer hilated by premultiplication by L_E There will be such matrices if the rank of L_E is less than six, for example, when ϕ is the mean square value of the magnitude of the error in the position estimate at the terminal time It may be shown, however, that all matrices K that result in $EH^T - KR$ being annihilated by L_E lead to the same value of ϕ The different possibilities for K result in different values of some of the elements of E_f which do not enter ϕ K from (3 13) is the only choice which is optimal for any B

Let us suppose, for this section, that $E(0)$ and $Q(t)$ are specified Eliminating K by (3 13), and defining

$$U = H^T R^{-1} H \tag{3 14}$$

we can reduce (3 11) to

$$\delta\phi = -\mathrm{tr}\int_0^{t_f} EL_E E\delta U\, dt \tag{3 15}$$

Since the coefficient of δU in (3 15) is not an explicit function of U, we cannot hope to find the optimal U in the same way we found the optimal K It is also clear that EL_EE can never be zero since E is nonsingular, and L_E cannot be zero as $\mathrm{tr}L_E\delta E$, at any time t, is the change in ϕ that would result from arbitrary $\delta E(t)$ if no changes in U were made over the remainder of the interval Further, any element of EL_EE can be zero over the interval only if a full row of $L_E(t_f)$ is zero This would mean that the estimation error in one of the state variables did not enter the problem at all Hence, certain elements of U would have no effect on ϕ and could be dropped from the problem

For any element of δU which has nonzero coefficient, however, it is clear that ϕ can always be decreased by a change in that element Thus, the measurement selection problem can only be meaningful if there are limits on U; either H must be limited, or there must be a constraining relation between $H(t)$ and $R(t)$ so as to keep every relevant element of $U(t)$ finite The minimum ϕ is obtained when there is no permissible δU for which $[-\mathrm{tr}EL_EE\delta U]$ is negative **

The variational Hamiltonian of this system is

$$\mathrm{tr}L_E\dot{E} = \mathrm{tr}L_EEF^T + \mathrm{tr}L_EFE - \mathrm{tr}EL_EEU + \mathrm{tr}L_EGQG^T \tag{3 16}$$

We can think of U as a matrix of "control" variables Since the Hamiltonian is linear in U, the maximum principle can be used to immediately state that every element of U will be on the boundary of its restricted set We conclude that maximizing $\mathrm{tr}EL_EEU$ with respect to U over the entire interval will produce minimum ϕ Inasmuch as E is given at the initial time and L_E at the final time, however, there is no direct way of finding the E, L_E, and H histories which maximize $\mathrm{tr}EL_EEU$ at each point in the interval

Characterization of the constraints on U is essential for numerical treatment of the problem of this section As U contains partial derivatives of the measurement functions and the covariance matrix of measurement errors, it is determined by the choice of the functions $M(x)$ The real control variables of the problem are the measurement choices Let us suppose that the measurement functions and their error-covariance matrices are determined by a k vector of parameters $\Theta(t)$ Then

$$\delta\phi = -\int_0^{t_f}\sum_{i\,j\,k}(EL_EE)_{ji}\frac{\partial U_{ij}}{\partial\Theta_k}\,\delta\Theta_k(t) \tag{3 17}$$

where $(\ )_{ij}$ is the ijth element of $(\ )$, Θ_k is the kth element of Θ, subscript notation being forced by the partial derivative of U_{ij} term By defining a vector

$$l_\Theta{}^T = -\sum_{i\,j}(EL_EE)_{ji}(\partial U_{ij}/\partial\Theta) \tag{3 18}$$

we can write

$$\delta\phi = \int_0^{t_f} l_\Theta{}^T\delta\Theta dt \tag{3 19}$$

In this form, the numerical optimization problem has received considerable attention The gradient methods introduced by Kelley and by Bryson appear well suited for successive improvements of the $\Theta(t)$ programs We may note that some elements of l_Θ may not explicitly depend on Θ In such cases there would necessarily be inequality constraints on the corresponding Θ elements Numerical solution of this singular problem, especially with multiple control variables, has as yet received little treatment in the literature

4 Feedback Gain History That Minimizes a Function of the Terminal Dispersion and the Control Used

In this section we assume that the navigation system has been specified Specifically, we mean that $U(t)$ is given and that $K(t)$ is given by (3 13) We now wish to find the feedback control gain matrix $\Lambda(t)$, which minimizes an ensemble average measure of a quadratic form in the terminal dispersions and the control used This performance index may be written

$$\phi = \mathcal{E}(\psi^TA\psi)_{t=t_f} + c\mathcal{E}\int_0^{t_f} f_0 dt \tag{4 1}$$

where A is given, f_0 is a scalar function of the control variables, and c is a chosen weighting factor An alternative version of this performance index, which is theoretically equivalent but numerically slightly harder, is to minimize $\mathcal{E}(\psi^TA\psi)_{t=t_f}$ while constraining the

$$\mathcal{E}\int_0^t f_0 dt$$

to be a given value We can use (4 1) as it stands then, but c becomes a Lagrange multiplier to be determined as part of the answer

By (2 3), the nominal value of $\psi[x(t_f)]$ is zero At $t = t_f$ then, to first order,

$$\psi = [(\partial\psi/\partial x)\delta x]_{t=t} \tag{4 2}$$

Hence,

$$\mathcal{E}(\psi^TA\psi)_{t=t_f} = \mathcal{E}\left[\delta x^T\left(\frac{\partial\psi}{\partial x}\right)^T A\,\frac{\partial\psi}{\partial x}\,\delta x\right]_{t=t_f}$$

$$= \mathrm{tr}\left[\left(\frac{\partial\psi}{\partial x}\right)^T A\,\frac{\partial\psi}{\partial x}\right]_{t=t_f} X(t_f) \tag{4 3}$$

where

$$X = \mathcal{E}(\delta x\,\delta x^T) \tag{4 4}$$

The matrix X is the ensemble average quadratic measure of the state perturbation This section is built primarily on it

We choose

$$L_X(t_f) = \left[\left(\frac{\partial\psi}{\partial x}\right)^T A\,\frac{\partial\psi}{\partial x}\right]_{t=t_f} \tag{4 5}$$

and write

$$\phi = \mathrm{tr}(L_XX)_{t=t_f} + c\mathcal{E}\int_0^{t_f} f_0 dt \tag{4 6}$$

To find necessary conditions for the extremalizing $\Lambda(t)$,

** Richard Moroney, consulting mathematician, contributed greatly to the authors' understanding of this problem The erroneous statements about the optimal $H(t)$ in the preprint (63-222) of this paper were due to misquoting Moroney's con clusions The first author apologizes to Moroney for badly using perfectly correct analysis we employ again the technique of Sec 3 First, with the same assumptions on v and w used in obtaining (3 7), we obtain

$$\dot{X} = (d/dt)\mathcal{E}(\delta x \delta x^T) = \mathcal{E}(d/dt)(\delta x \delta x^T) = (F - G\Lambda)X + X(F - G\Lambda)^T + G\Lambda(E - N) + (E - N^T)\Lambda^T G^T + GQG^T \quad (4\,7)$$

where

$$N = \mathcal{E}[\delta\hat{x}(\delta\hat{x} - \delta x)^T] \quad (4\,8)$$

But we can also obtain

$$\dot{N} = (F - G\Lambda)N + N(F - KH)^T + KHE - KRK^T \quad (4\,9)$$

For $K = EH^TR^{-1}$, the last two terms in (4 9) cancel, leaving N homogeneous in N Hence, if $N(0) = 0$, as we shall assume here, $N(t) = 0$ Thus we write (4 7) as

$$\dot{X} = (F - G\Lambda)X + X(F - G\Lambda)^T + G\Lambda E + E\Lambda^T G^T + GQG^T \quad (4\,10)$$

Since the measurement program and the estimator gain matrix are assumed to be given here, the $E(t)$ history may be calculated directly from (3 7) Then, a nominal selection of $\Lambda(t)$ will lead to a nominal $X(t)$ history Small variations in $X(t)$ will satisfy [with $\delta Q(t) = 0$]

$$\delta\dot{X} = (F - G\Lambda)\delta X + \delta X(F - G\Lambda)^T - (X - E)\delta\Lambda^T G^T - G\delta\Lambda(X - E) \quad (4\,11)$$

We must also consider

$$\delta\mathcal{E}\int_0^{t_f} f_0 dt = \delta\int_0^{t_f}\mathcal{E}(f_0)dt$$

We take $\mathcal{E}(f_0)$ to be any function of the control that can be written as the sum of matrix products involving Λ, X, N, and E Two major examples will be given Using the chain rule of differentiation and the trace properties, we obtain

$$\delta\mathcal{E}(f_0) = \mathrm{tr}A_X\delta X + \mathrm{tr}A_\Lambda\delta\Lambda \quad (4\,12)$$

for appropriate definitions of A_X and A_Λ

The governing differential equation for L_X is chosen as

$$\dot{L}_X = -(F - G\Lambda)^T L_X - L_X(F - G\Lambda) - cA_X \quad (4\,13)$$

so that

$$(d/dt)\,\mathrm{tr}(L_X\delta X) + c\,\mathrm{tr}A_X\delta X + c\,\mathrm{tr}A_\Lambda\delta\Lambda = \mathrm{tr}[-2(X - E)L_XG + cA_\Lambda]\delta\Lambda \quad (4\,14)$$

Integration of (4 14) from $t = 0$ to $t = t_f$ gives [with $\delta X(0) = 0$]

$$\delta\phi = \mathrm{tr}\int_0^{t_f}[-2(X - E)L_XG + cA_\Lambda]\delta\Lambda dt \quad (4\,15)$$

In order that ϕ be a minimum with respect to arbitrary (small) variations $\delta\Lambda$, the usual necessary condition is that

$$A_\Lambda = (2/c)(X - E)L_XG \quad (4\,16)$$

If A_Λ is a function of Λ, (4 16) determines $\Lambda(t)$ for minimum ϕ As usual, we have a two-point boundary-value problem with $X(0)$ specified and $L_X(t_f)$ given by (4 5)

As one special case, consider that $f_0 = u^TWu$, a quadratic form in the control used, where $W_u(t)$ is a chosen symmetric matrix We write

$$\mathcal{E}(f_0) = \mathcal{E}(u^TW_uu) = \mathcal{E}(\delta\hat{x}^T\Lambda^TW_u\Lambda\delta\hat{x}) = \mathrm{tr}\mathcal{E}(\delta\hat{x}\delta\hat{x}^T\Lambda^TW_u\Lambda) = \mathrm{tr}(X - E)\Lambda^TW_u\Lambda \quad (4\,17)$$

by using $N = 0$ We obtain

$$A_\Lambda = 2(X - E)\Lambda^TW_u \quad (4\,18)$$

which, from (4 16), gives (observing that L_X is symmetric) the optimal Λ as

$$\Lambda = (1/c)W_u^{-1}G^TL_X \quad (4\,19)$$

This result is particularly interesting in that Λ does not depend on X †† $L_X(t)$ can be calculated by integration of (4 13) from the terminal point, substituting for Λ from (4 19) Then $\Lambda(t)$ can be calculated directly from (4 19) What is normally a two-point boundary value problem has split into two one-point boundary-value problems if L_X and then X is calculated

An important paper by Striebel and Breakwell[5] has considered a problem with a scalar control in which the statistical measure of control used is the expected value of the integral absolute value control (deviation) used An assumption of considerable plausibility in their analysis is that the form of Λ is given by

$$\Lambda = k\lambda^T \quad (4\,20)$$

where $\lambda(t)$ satisfies

$$\dot{\lambda} = -F^T\lambda \quad (4\,21)$$

with boundary conditions given at t_f, and $k(t)$ is a scalar gain In our terms, they have chosen

$$\mathcal{E}(f_0) = \mathcal{E}|u| = \mathrm{const}[k\lambda^T(X - E)\lambda k]^{1/2} \quad (4\,22)$$

Inasmuch as k appears linearly in f_0, A_Λ will not contain k, and (4 16) will not give a solution for k This is a singular problem in the calculus of variations The solution will, in general, involve portions using the limits on k and singular arcs along which k is determined from the (in nearly all cases) second time derivative of (4 16) In Ref 5, with $Q = 0$, the singular arc is found by an ingenious application of Greene's theorem We note that assumption (4 20) does not cause the singular nature of the problem The cause is the form of $\mathcal{E}(f_0)$ as given in (4 22)

For $f_0 = u^TWu$, the optimal Λ may be numerically determined as described For the scalar control, scalar gain, $f_0 = |u|$ of (4 22), we anticipate a singular arc solution similar to that of Ref 5, even with $Q \neq 0$ In this case it appears that the approach of Ref 5 still could be successfully used, although a family of singular arcs would be needed in properly matching the $\Lambda = 0$ intervals to the appropriate singular arc Calculation of the optimal $\Lambda(t)$ for $f_0 = [u^TWu]^{1/2}$ with vector u promises to be a more difficult problem The gradient method has not been sufficiently proved for such singular problems to guarantee acceptable convergence Other functions for f_0, which we might admit, seem less significant, and we shall defer further discussion of numerical solution for continuous $\Lambda(t)$ until Sec 6

5 Optimal Spacing of Impulsive Velocity Corrections

In Sec 4 the measurements and the feedback control were both assumed continuous Under this assumption, and using the estimator gain of (3 13), the covariance matrix $E(t)$ may be calculated before the dispersion matrix $X(t)$ In this section we still assume continuous measurements, but take the feedback control to be a series of impulsive (essentially instantaneous) velocity corrections Assuming a fixed number of corrections, we wish to find the times of the corrections to minimize the same performance index as in Sec 4

We may employ the state perturbation transition matrix to write

$$\delta x(t) = \Phi(t,\tau)\delta x(\tau) \quad (5\,1)$$

if there has been no correction between τ and t The esti-

†† This result is now widely known and may be derived with out resort to the matrix adjoint

mator, assuming continuous measurements, is still (2 5) There is, however, no $u(t)$ except at the discrete correction times

At a correction time t_i we have

$$\delta x(t_i+) = \delta x(t_i-) + G(t_i)(\Delta_u + \Delta_v)_{t=t_i} \quad (5\,2)$$

$$\delta \hat{x}(t_i+) = \delta \hat{x}(t_i-) + G(t_i)\Delta_u(t_i) \quad (5\,3)$$

where

$$\Delta_u(t_i) = \int_{t_i-}^{t_i+} u(t)dt \qquad \Delta\ (t_i) = \int_{t_i-}^{t_i+} v(t)dt$$

In effect, we choose $\Lambda(t)$ to be zero everywhere but at the t_i, where it is (mathematically) infinite We write this as

$$\Delta_u(t_i) = -\Gamma(t_i)\delta\hat{x}(t_i-) \quad (5\,4)$$

where Γ is a 3×6 matrix, making Δ_u a three-component velocity impulse As assumed by Battin,[2] $\Gamma(t)$ is the precalculated matrix which will make the terminal position deviation predicted by $\Phi(t_f, t)\delta\hat{x}(t+)$ equal to zero This choice of Δ_u keeps $N(t) = 0$, as Battin showed

We pick the performance index (4 6) with (4 17), which becomes, in this section,

$$\phi = \mathrm{tr}(L_X X)_{t=t_f} + c\sum_{i=1}^{s} \mathcal{E}[\Delta_u{}^T W_u \Delta_u]_{t=t_i}$$

$$= \mathrm{tr}(L_X X)_{t=t_f} + c\sum_{i=1}^{s} \mathrm{tr}[W\ \Gamma(X - E)\Gamma^T]_{t=t_i-} \quad (5\,5)$$

where s velocity corrections are made To evaluate ϕ, then, we need the values of X and E at each t_i-, $i = 1, 2, \quad , s$ Between corrections we may use (5 1) to obtain directly

$$X(t_{i+1}-) = \Phi(t_{i+1}-,t_i+)X(t_i+)\Phi^T(t_{i+1}-,t_i+) \quad (5\,6)$$

At a correction time we use (5 3) and (5 4) to obtain

$$X(t_j+) = X(t_j-) + G(t_j)[\Gamma(X - E)\Gamma^T]_{t=t_j-}\ G^T(t_j) + G(t_j)Q_jG^T(t_j) + [(X - E)\Gamma^TG^T + G\Gamma(X - E)]_{t=t_j-} \quad (5\,7)$$

where, based on the v and w assumptions of (3 7),

$$[\Delta_u(t_j)\Delta_u{}^T(t_j)] = \Gamma(t_j)\mathcal{E}(\delta\hat{x}\delta\hat{x}^T)_{t=t_j-}\ \Gamma^T(t_j) = \Gamma(t_j)(X - E)_{t=t_j-}\ \Gamma^T(t_j) \quad (5\,8)$$

$$Q_j = \mathcal{E}[\Delta\ (t_j)\Delta_v{}^T(t_j)] = \mathcal{E}\left[\int_{t_j-}^{t_j+} v(\tau)d\tau \int_{t_j-}^{t_j+} v^T(\tau')d\tau'\right] = \int_{t_j-}^{t_j+} Q(\tau)d\tau \quad (5\,9)$$

In taking the limit of (5 9) as $t_j+ \rightarrow t_j-$ we see that Q_j will approach zero unless $Q(t)$ is infinite at $t = t_j$ This mathematical limit of $Q(t)$ to obtain finite Q_j is fully consistent with the limit of impulsive velocity correction to obtain finite Δ As a practical matter, where $u(t)$ is bounded, $Q(t)$ would more likely be estimated on the basis of experimentally obtained Q_j and the small but finite length of time required to make the correction

In this section in particular, the assumption that control-implementation error depends only upon time appears questionable As Battin[2] has assumed, the implementation error is more likely also (or only) to depend (statistically) on the control itself This more general situation will be included in the next section It makes the mutual coupling of $X(t)$ and $E(t)$ more explicit than in this section

By assuming continuous measurements,‡‡ and using $K(t)$ from (3 13) and $Q(t) = 0$, (3 7) becomes, between corrections,

$$\dot{E} = FE + EF^T - EH^TR^{-1}HE \quad (5\,10)$$

‡‡ The case of a discrete sequence of measurements is treated in the numerical example of Sec 7

At correction time t_j, E has a jump given by

$$E(t_j+) = E(t_j-) + G(t_j)Q_jG^T(t_j) \quad (5\,11)$$

Note that, even with Q_i, $i = 1, 2, \quad s$, specified, $E(t)$ cannot be determined until the correction times t_i are known (or, in other words, until $\Lambda(t)$ is known) This is in contrast to the previous two sections in which $E(t)$ could be calculated independent of the feedback control

The problem is now one of optimizing a set of parameters, namely, the correction times t_i, $i = 1, 2, \quad , s$ Conceptually, to minimize ϕ it is only necessary to set

$$\partial\phi/\partial t_i = 0 \qquad i = 1, 2, \quad , s \quad (5\,12)$$

Practically, a successive approximation technique will be required, using $\partial\phi/\partial t_i$ determined numerically The simplest scheme would be to pick a set of values for the t_i, then in turn, vary each one a small amount §§ Using $t_j + \Delta t_j$ in place of t_j, with all other t_i the same, the performance index would be calculated as $\phi_{\mathrm{nom}} + \Delta_j\phi$ Then we would approximate

$$\partial\phi/\partial t_j \approx \Delta_j\phi/\Delta t_j \quad (5\,13)$$

The successive improvement technique starts with nominal choices of the t_i, with corresponding $X(t)$, $E(t)$, and ϕ The partial derivatives (5 12) are evaluated, and then changes are made in the t_i in the direction of decreasing ϕ

6 Simultaneous Optimization of Continuous Measurement and Feedback Gain Programs

If both the measurement program and the feedback control gain program are open to be optimized, they must be considered together $\Theta(t)$, with $K(t)$, dictates $E(t)$ which enters into $\dot{X}(t)$ The optimal $\Theta(t)$, $K(t)$, and $\Lambda(t)$ are thus implicitly coupled Coupling also occurs if the control implementation error (statistically) depends on the control and/or on the state This may bring $\Lambda(t)$ directly into the $\dot{E}$ equation

Since it has been convenient to deal with X and E thus far, we shall retain these and introduce the cross-product correlation matrix

$$Y = \mathcal{E}(\delta x\ e^T) \quad (6\,1)$$

We then can write

$$\begin{pmatrix} X & Y \\ Y^T & E \end{pmatrix} = \mathcal{E}\left[\begin{pmatrix} \delta x \\ e \end{pmatrix}(\delta x^T\ e^T)\right] \quad (6\,2)$$

$\dot{E}$ is taken from (3 7), $\dot{X}$ from (4 7) with N eliminated by the substitution $N = E + Y$ $\dot{Y}$ is

$$\dot{Y} = (d/dt)\mathcal{E}[\delta x\ e^T] = \mathcal{E}[\delta\dot{x}\ e^T + \delta x\ \dot{e}^T] = (F - G\Lambda)Y + Y(F - KH)^T - G\Lambda E - GQG^T \quad (6\,3)$$

We generalize the performance indices of Secs 3–5 by letting ϕ be any sum of matrix products involving X, Y, and E (and, if desired, their inverses) at the terminal point plus an integral (statistical) measure of the control used The variation of ϕ then satisfies

$$\delta\phi = [\mathrm{tr}L_X\ \delta X + \mathrm{tr}L_{Y^T}\ \delta Y + \mathrm{tr}L_Y\ \delta Y^T + \mathrm{tr}L_E\ \delta E]_{t=t_f} + c\int_0^{t_f} \{\mathrm{tr}[A_X\delta X + A_Y{}^T\delta Y + A_Y\ \delta Y^T + A_E\delta E] + \mathrm{tr}A_\Lambda\delta\Lambda\}dt \quad (6\,4)$$

where L_X, L_Y, and L_E at $t = t_f$ depend upon the definition of ϕ They would be functions of X, Y, and E if ϕ were more

§§ Fitzgerald[6] of Massachusetts Institute of Technology offers a method which may not save computation time relative to the simple approach given here, but which should provide greater accuracy in obtaining $\partial\phi/\partial t_i$

complicated than a linear function of the elements of X, Y, and E Ordinarily, however, ϕ as a linear function is sufficient if the ensemble average quadratic functions are adequate for measuring the performance

Let us pick an example problem that requires the more detailed analysis of this section The performance index is essentially taken from Sec 4:

$$\phi = \mathcal{E}\{\psi^T[x(t_f),t_f]A\psi[x(t_f),t_f]\} + \mathcal{E}\int_0^{t_f} u^T W u dt \quad (6\ 5)$$

where, however, the terminal time is not specified We must identify the criterion that determines the terminal point on each member of the ensemble Our development here is using the space mission midcourse guidance problem for backdrop The transition from midcourse to terminal phase will typically be voluntary, which is to say the terminal phase is initiated on the basis of the estimate of the state The midcourse phase is terminated when the estimate of a certain function of the state and time reaches zero¶¶:

$$\Omega[\hat{x}(t),t]_{t=t_f} = 0 \quad (6\ 6)$$

The first term in the performance may be expanded as

$$\mathcal{E}(\psi^T A \psi)_{t=t_f} = \mathcal{E}[(\psi_{\text{nom}} + d\psi)^T A(\psi_{\text{n m}} + d\psi)]_{t=t_f}$$
$$= \mathcal{E}[d\psi^T A d\psi]_{t=t_f} \quad (6\ 7)$$

because $\psi_{\text{nom}} = 0$ But $d\psi$ is the deviation from $\psi = 0$ due to perturbations from the nominal trajectory Thus, it may be written as

$$d[\psi(x_f,t_f)] = \delta\psi(t_f) + \dot{\psi} dt_f \quad (6\ 8)$$

where

$$\dot{\psi} = [(\partial\psi/\partial x)\dot{x} + (\partial\psi/\partial t)]_{t=t_f}$$

is evaluated on the nominal path But since $\Omega[\hat{x}_f,t_f] = 0$ on each trajectory, we have

$$d[\Omega(\hat{x}_f,t_f)] = \delta\Omega(t_f) + \dot{\Omega} dt_f = 0 \quad (6\ 9)$$

where

$$\dot{\Omega} = [(\partial\Omega/\partial\hat{x})]\dot{x} + (\partial\Omega/\partial t)]_{t=t}$$

and we have used $\dot{\hat{x}} = \dot{x}$ on the nominal Solving (6 9) for dt_f, we can write

$$d[\psi(x_f,t_f)] = \delta\psi(t_f) - \frac{\dot{\psi}}{\dot{\Omega}}\,\delta\Omega(t_f)$$
$$= \left[\frac{\partial\psi}{\partial x}\,\delta x - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\,\delta\hat{x}\right]_{t=t_f}$$
$$= \left[\left(\frac{\partial\psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)\delta x + \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\,e\right]_{t=t_f} \quad (6\ 10)$$

after substituting for $\delta\hat{x}$ Now

$$\mathcal{E}(\psi^T A\psi)_{t=t_f} = \mathcal{E}\Bigg\{\text{tr}\left[\left(\frac{\partial\psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)^T A\left(\frac{\partial\psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)\times \delta x\delta x^T\right] + \text{tr}\left[\left(\frac{\partial\psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)^T A\left(\frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)e\delta x^T\right] +$$
$$\text{tr}\left[\left(\frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)^T A\left(\frac{\partial\psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)\delta x\, e^T\right] + \text{tr}\left[\left(\frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)^T A\left(\frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial\hat{x}}\right)ee^T\right]\Bigg\}_{t=t_f} \quad (6\ 11)$$

Note that this portion of the performance index involves X, Y, Y^T, and E at $t = t_f$

¶¶ The independent variable is assumed precisely known here This assumption can be removed by adding t as an additional state variable to be included in the estimator

Now consider the integral in the performance index We have

$$\mathcal{E}(u^T W u) = \mathcal{E}[\delta\hat{x}^T\Lambda^T W\Lambda\delta\hat{x}]$$
$$= \mathcal{E}[(\delta x^T + e^T)\Lambda^T W\Lambda(\delta x + e)]$$
$$= \text{tr}\Lambda^T W\Lambda(X + Y + Y^T + E) \quad (6\ 12)$$

The integrand for this example thus contains Λ, X, Y, Y^T, and E

To complete the specification of the example problem, let us suppose that $H(t)$ and $R(t)$ are specified, but that the covariance of control implementation error is proportional to the covariance of the control:

$$Q = \sigma^2\mathcal{E}(uu^T) = \sigma^2\mathcal{E}(\Lambda\delta\hat{x}\,\delta\hat{x}^T\Lambda^T)$$
$$= \sigma^2\Lambda^T\Lambda(X + Y + Y^T + E) \quad (6\ 13)$$

The initial conditions $X(0)$, $Y(0)$, and $E(0)$ are given The problem is to choose $\Theta(t)$, $K(t)$, and $\Lambda(t)$ to minimize (6 5)

To carry out the analysis for the general problem of this section we need the following perturbation equations:

$$\delta\dot{X} = (F - G\Lambda)\delta X + \delta X(F - G\Lambda)^T - G\Lambda\delta Y^T - \delta Y\Lambda^T G^T - G\delta\Lambda X - X\delta\Lambda^T G^T - G\delta\Lambda Y^T - Y\delta\Lambda^T G^T + G\delta Q G^T \quad (6\ 14)$$

$$\delta\dot{Y} = (F - G\Lambda)\delta Y + \delta Y(F - KH)^T - G\Lambda\delta E - G\delta\Lambda E - G\delta\Lambda Y - YH^T\delta K^T - Y\delta H^T K^T - G\delta Q G^T \quad (6\ 15)$$

$$\delta\dot{E} = (F - KH)\delta E + \delta E(F - KH)^T - \delta KHE - K\delta HE - EH^T\delta K^T - E\delta H^T K^T + \delta KRK^T + K\delta RK^T + KR\delta K^T + G\delta Q G^T \quad (6\ 16)$$

In this section we wish to explicitly allow Q to be a function of X, Y, E, and Λ as well as t We write

$$\delta Q = Q_X\delta X + Q_{Y^T}\delta Y + Q_Y\delta Y^T + Q_E\delta E + Q_\Lambda\delta\Lambda \quad (6\ 17)$$

We choose

$$\dot{L}_X = -L_X(F - G\Lambda) - (F - G\Lambda)^T L_X - G^T(L_X - L_{Y^T} - L_Y + L_E)GQ_X - cA_X \quad (6\ 18)$$

$$\dot{L}_{Y^T} = -L_{Y^T}(F - G\Lambda) - (F - KH)^T L_{Y^T} + \Lambda^T G^T L_X - G^T(L_X - L_{Y^T} - L_Y + L_E)GQ_Y - cA_{Y^T} \quad (6\ 19)$$

$$\dot{L}_E = L_{Y^T}G\Lambda + \Lambda^T G^T L_Y - L_E(F - KH) - (F - KH)^T L_E - G^T(L_X - L_{Y^T} - L_Y + L_E)GQ_E - cA_E \quad (6\ 20)$$

to yield (with $\delta X(0) = \delta Y(0) = \delta E(0) = 0$)

$$\delta\phi = \int_0^{t_f}\left\{\begin{array}{l} -2\text{tr}[(X + Y)L_X G + (E + Y)L_{Y^T}G - G^T(L_X - L_{Y^T} - L_Y + L_E)GQ_\Lambda - c\,A_\Lambda]\delta\Lambda \\ -2\text{tr}[K^T L_{Y^T} Y + K^T L_E E]\delta H^T \\ +\ \text{tr}K^T L_E K\,\delta R \\ -2\ \text{tr}[L_{Y^T}\,YH^T + L_E EH^T - L_E KR]\delta K^T \end{array}\right\} dt \quad (6\ 21)$$

The estimator gain, if ϕ is to be stationary must satisfy

$$K = L_E^{-1}(L_{Y^T}Y + L_E E)H^T R^{-1} \quad (6\ 22)$$

The feedback control gain, if A_Λ contains Λ, must satisfy

$$A_\Lambda = (1/c)[(X + Y)L_X + (E + Y)L_{Y^T}]G - (1/c)G^T(L_X - L_{Y^T} - L_Y + L_E)GQ_\Lambda \quad (6\ 23)$$

To treat the δH^T and δR terms, we again assume both $H(t)$ and $R(t)$ to be parametrically determined by a vector $\Theta(t)$

Table 1 Nominal measurement program obtained by Battin's optimization procedure

Measurement time, hr	Parameter values defining the measurements	Mean square position uncertainty, miles²	Mean square velocity uncertainty, (mph)²	Response function $\lvert\lambda_n^{(1)}\rvert$, (miles³/rad) × 10² a[d]	b[d]
0		12 538	127 630		
0 6	9[a] n[b] e[c]	15.422	101 454	0 102	0 237
0 9	19 *ne*	16 782	64 946	0 367	0 792
1 2	9 *ne*	24 898	47 341	0 100	0 183
1 5	5 *ne*	21 607	30 503	0 133	0 294
1 8	18 *ne*	30 822	26 239	0 212	0 461
2 2	5 *ne*	42 826	21 539	0 898	0.216
2 6	15 *ne*	48 806	17 284	0 207	0 426
3 0	15 *fe*	62 672	15 654	0 197	0 402
3 4	15 *ne*	70 659	13 150	0 146	0 296
3 8	15 *ne*	84 590	12 115	0 120	0 244
4 5	15 *ne*	118 943	11 530	0 083	0 171
5 5	15 *ne*	176 184	10 859	0 047	0 104
6 0	16 *nm*	195 148	9 784	0 016	0 034
6 5	15 *ne*	208 949	8 655	0 027	0 069
7 0	16 *nm*	217 329	7 452	0 011	0 022
7 5	16 *nm*	225 964	6 480	0 009	0 016
8 5	15 *ne*	264 946	5 678	0 012	0 025
9 5	17 *fm*	288 075	4 590	0 006	0 008
10 0	17 *fm*	284 664	3 835	0 005	0 006
10 5	15 *ne*	288 146	3 376	0 023	0 052
12 0	17 *fm*	335 194	2 769	0 018	0 063
12 5	17 *fm*	333 797	2 371	0 025	0 084
13 5	15 *ne*	363 928	2 131	0 081	0 204
15 0	17 *fm*	406 820	1 772	0 074	0 226
16 0	17 *fm*	430 987	1 539	0 101	0 302
17 0	15 *ne*	460 187	1 404	0 177	0 432
19 5	17 *fm*	547 520	1 162	0 234	0 667
22 0	17 *fm*	637 704	0 981	0 369	1 034
23 5	15 *ne*	686 027	0 893	0 340	0 914
28 5	17 *fm*	876 648	0 685	0 925	2 521
29 5	14 *fe*	252 091	0 283	0 430	2 753
37 0	17 *fm*	300 406	0 175	2 341	6 285
40 5	20 *ne*	301 103	0 193	1 295	3 435
53 5	17 *fm*	361 114	0 059	12 296	32 821
57 5	17 *fm*	362 827	0 046	18 707	50 302
60 0	17 *fm*	323 575	0 200	19 388	53 400
60 5	3 *nm*	17 272	0 142	17 680	79 844
61 4	15 *nm*	3 578	0 124	9 023	27 892
62 1	5 *nm*	0 939	0 361	7 107	14 847
62 4	7 *nm*	0 582	1 993	4 464	7 552
62 56		0 560	5 031		

[a] The number of the star sighted The numbers (in order of brightness) are: 1) αC Major (Dirius) 2) α Cardinal (Canopus) 3) α Centauri 4) α Lyral (Vega) 5) α Aurigae (Capella) 6) α Bootis (Acturis) 7) β Orionis (Riegel), 8) αC Minor (Procyon) 9) α Eridani (Achernar) 10) Orionis (Betelgeuse) 11) β Centauri 12) α Aquelae 13) α Crucis m, 14) α Tauri (Aldebaran), 15) β Geminarium (Pollux) 16) α Virginis (Speca) 17) α Scorpii (Antares), 18) α Piscis Aust (Formalhaust) 19) α Cygni (Deneb) and 20) α Leonis (Regulus)
[b] *n* or *f* referring to the near or far body horizon sighted
[c] Earth (*e*) or moon (*m*) horizon sighted
[d] Response function for a) terminal mean square position uncertainty; and b) weighted sum of terminal mean square position and velocity uncertainties

We write, defining m_Θ,

$$\sum_{i\,j\,k\,p\,q}\left[-2(K^T L_Y^T Y + K^T L_E E)_{ji}\frac{\partial H_{ij}^T}{\partial \Theta_k} + (K^T L_E K)_{pq}\frac{\partial R_{qp}}{\partial \Theta_k}\right]\delta\Theta_k = m_\Theta^T\,\delta\Theta \qquad (6\ 24)$$

Setting m_Θ equal to zero will then determine Θ if $\partial m_\Theta/\partial\Theta$ is a nonsingular matrix If this fails, the singular variational problem is again obtained In either case, $\Theta(t)$ must be chosen so that $m_\Theta^T\delta\Theta(t)$ is positive or zero for all permissible $\delta\Theta(t)$

The optimal estimator gain for the coupled problem of this section is not, in general, the same as for the navigation problem Also, even if $f_0 = u^T W_u u$, the optimal feedback control gain matrix is not, in general, the same as it would be with perfect measurements These statements apply even for specified $\Theta(t)$

Bryson[7] has shown the authors that the results of Secs 3 and 4 are jointly correct here for the case in which

$$\left.\begin{array}{c} \Theta(t) \text{ is specified} \\ Y(0) + E(0) = 0\ [N(0) = 0] \\ L_Y(t_f) = 0[L_X(t_f) \text{ and } L_E(t_f) \text{ const}] \\ f_0 = [\delta x^T \delta u^T]\begin{bmatrix} W_{xx} & W_{xu} \\ W_{ux} & W_{uu}\end{bmatrix}\begin{bmatrix} \\ \delta u\end{bmatrix} \\ Q_X = Q_Y = Q_E = Q_\Lambda = 0 \end{array}\right\} \qquad (6\ 25)$$

Giving (6 25) leads to

$$\begin{array}{c} Y(t) + E(t) = 0 \\ L_Y(t) = 0 \\ K = EH^T R^{-1} \\ \Lambda = (1/c)W_{uu}^{-1}(G^T L_X - W_{ux}) \end{array} \qquad (6\ 26)$$

In this case $K(t)$ and $E(t)$ can be calculated directly Then, Λ in (6 18) may be eliminated by using (6 26) L_X from the integration of (6 18) backwards from $t = t_f$ then gives $\Lambda(t)$ from (6 26) The split of the two-point boundary-value problems into one-point boundary-value problems occurs just as it did in Sec 4

Unless (6 25) holds, however, the two-point boundary-value problem, X, Y, E given at $t = 0$, and L_X, L_Y, L_E given at $t = t_f$, must be solved by an iterative technique We shall now outline the steepest-ascent technique of Refs 3 and 4 for the case in which m_Θ is explicitly a function of Θ such that $m_\Theta = 0$ would uniquely determine Θ and, similarly, in which (6 23) would uniquely determine Λ We write (6 21) as

$$\delta\phi = \int_0^{t_f} [\mathrm{tr}\, \mathcal{L}_K{}^T(\tau)\delta K(\tau) + \mathrm{tr}\mathcal{L}_\Lambda{}^T(\tau)\delta\Lambda(\tau) + m_\Theta{}^T(\tau)\delta\Theta(\tau)]d\tau \quad (6\ 27)$$

where $\mathcal{L}_K$, $\mathcal{L}_\Lambda$, and m_Θ are matrices evaluated using nominal functions K, Λ, and Θ They may be identified from (6 21) and (6 24) The object is to choose $\delta K(t)$, $\delta\Lambda(t)$, and $\delta\Theta(t)$ to decrease ϕ as much as possible while keeping δX, δY, and δE small enough for the linearized perturbation equations to hold There is no technique for directly achieving this The common version of steepest-ascent minimizes $\delta\phi$ for a given choice of

$$(dP)^2 = \int_0^{t_f} [\mathrm{tr}\delta K^T W_K \delta K + \mathrm{tr}\delta\Lambda^T W_\Lambda \delta\Lambda + \delta\Theta^T W_\Theta \delta\Theta]dt \quad (6\ 28)$$

where dP is the measure of change in the "control" variables and is picked experimentally at a level which "stretches" but does not seriously violate the linearity The weighting (symmetric) matrices W_K, W_Λ, W_Θ are chosen experimentally to aid convergence of the technique Their choices can be, numerically, very important in some cases The optimal choices of δK, $\delta\Lambda$, and $\delta\Theta$ are

$$\begin{aligned} \delta K(t) &= -W_K{}^{-1}(t)\mathcal{L}_K(t)(dP/J^{1/2}) \\ \delta\Lambda(t) &= -W_\Lambda{}^{-1}(t)\mathcal{L}_\Lambda(t)(dP/J^{1/2}) \\ \delta\Theta(t) &= -W_\Theta{}^{-1}(t)m_\Theta(t)(dP/J^{1/2}) \end{aligned} \quad (6\ 29)$$

where

$$J = \int_0^{t_f} [\mathrm{tr}\mathcal{L}_K{}^T W_K{}^{-1}\mathcal{L}_K + \mathrm{tr}\mathcal{L}_\Lambda{}^T W_\Lambda{}^{-1}\mathcal{L}_\Lambda + m_\Theta{}^T W_\Theta{}^{-1} m_\Theta]dt$$

As the minimum ϕ is approached, $\mathcal{L}_K$, $\mathcal{L}_\Lambda$, and m_Θ must each approach zero J thus measures the gradient before each iteration In fact, the predicted improvement with (6 29) is

$$\delta\phi = -dP\ J^{1/2} \quad (6\ 30)$$

As J becomes smaller it is clear that the rate of improvement will decrease The strength of the gradient method is that large changes in $\delta\phi$ can be obtained when J is large Convergence to the neighborhood of the optimal programs is rapid, although achieving extremely close approximations to exact extremals is likely to be costly

Table 2 Measurement program for the minimum terminal mean square position uncertainty

Measurement time, hr	Parameter values defining the measurements	Mean square position uncertainty, miles²	Mean square velocity uncertainty, (mph)²	Response function $\|\lambda_n{}^{(1)}\|$, (miles³/rad) × 10²
0		12 538	127 630	
0 6	4 *n e*	15 256	100 022	0 045
0 9	9 *ne*	16 043	64 237	0 078
1 2	7 *fe*	24 683	49 774	0 102
1 5	18 *ne*	34 464	43 277	0 120
1 8	1 *ne*	32 219	27 563	0 105
2 2	5 *fe*	49 077	26 393	0 105
2 6	7 *fe*	59 492	21 500	0 093
3 0	7 *ne*	65 556	17 100	0 081
3 4	15 *fe*	82 262	16 700	0 076
3 8	15 *fe*	105 552	16 903	0 065
4 5	10 *fe*	173 431	19 095	0 052
5 5	5 *fe*	267 436	19 304	0 034
6 0	5 *fe*	323 979	19 487	0 028
6 5	5 *fe*	393 138	19 970	0 023
7 0	14 *ne*	388 370	16 885	0 021
7 5	14 *ne*	447 021	16 806	0 016
8 5	5 *fe*	564 891	16 362	0 011
9 5	5 *fe*	712 159	16 374	0 008
10 0	15 *fe*	435 990	9 117	0 014
10 5	15 *fe*	391 765	7 445	0 016
12 0	15 *fe*	462 998	6 697	0 024
12 5	15 *fe*	451 804	6 021	0 028
13 5	15 *fe*	494 630	5 636	0 035
15 0	15 *fe*	578 038	5 316	0 048
16 0	15 *fe*	608 443	4 910	0 058
17 0	15 *fe*	638 830	4 561	0 068
19 5	15 *fe*	796 687	4 314	0 096
22 0	15 *fe*	922 486	3 923	0 127
23 5	15 *fe*	934 006	3 481	0 146
28 5	13 *fm*	1512 507	3 871	0 140
29 5	13 *nm*	1662 844	3 986	0 162
37 0	13 *fm*	2503 577	3 873	0 394
40 5	13 *fm*	2586 200	3 373	0 569
53 5	13 *fm*	896 759	0 758	2 190
57 5	20 *fm*	56 574	0 068	5 724
60 0	20 *fm*	54 274	0 266	6 482
60 5	6 *nm*	35 920	0 311	3 689
61 4	20 *fm*	24 668	1 070	4 281
62 1	5 *nm*	0 708	0 296	4 700
62 4	7 *nm*	0 427	1 323	3 685
62 56		0 456	3 140	

Table 3 Measurement program for the minimum weighted sum of terminal mean square position and velocity uncertainties

Measurement time, hr	Parameter values defining the measurements	Mean square position uncertainty, miles^2	Mean square velocity uncertainty, (mph)^2	Response function $\|\lambda_n^{(1)}\|$, (miles^3/rad) $\times 10^2$
0		12 538	127 630	
0 6	19 *ne*	16 037	105 594	0 121
0 9	9 *ne*	16 622	64 079	0 196
1 2	19 *ne*	25 525	47 660	0 241
1 5	18 *ne*	32 307	40 400	0 260
1 8	7 *ne*	30 747	26 243	0 253
2 2	7 *fe*	41 408	21 351	0 248
2 6	7 *fe*	63 316	21 862	0 227
3 0	7 *ne*	66 632	16 169	0 201
3 4	15 *fe*	69 837	14 155	0 155
3 8	15 *fe*	90 850	14 587	0 131
4 5	5 *fe*	144 797	16 098	0 119
5 5	5 *fe*	250 783	18 071	0 084
6 0	5 *fe*	313 426	18 769	0 069
6 5	5 *fe*	385 191	19 463	0 057
7 0	5 *fe*	466 229	20 139	0 046
7 5	5 *fe*	557 079	20 806	0 036
8 5	5 *fe*	781 313	22 439	0 023
9 5	5 *fe*	1056 060	24 056	0 015
10 0	15 *fe*	485 187	10 123	0 021
10 5	15 *ne*	270 270	5 126	0 027
12 0	7 *fe*	359 490	5 178	0 008
12 5	7 *fe*	383 359	5 081	0 010
13 5	15 *fe*	407 962	4 641	0 062
15 0	15 *fe*	480 955	4 425	0 088
16 0	8 *fe*	546 937	4 424	0 104
17 0	8 *fe*	619 208	4 435	0 126
19 5	8 *fe*	850 494	4 624	0 188
22 0	16 *fm*	152 815	0 614	0 245
23 5	16 *fm*	104 725	0 351	0 313
28 5	16 *fm*	104 506	0 223	0 631
29 5	13 *nm*	107 464	0 222	0 539
37 0	13 *fm*	185 917	0 257	1 266
40 5	13 *fm*	230 867	0 274	1 805
53 5	13 *fm*	369 212	0 309	6 676
57 5	13 *fm*	228 773	0 254	10 124
60 0	20 *fm*	80 504	0 403	16 851
60 5	6 *nm*	39 729	0 345	11 961
61 4	20 *fm*	30 888	0 135	11 367
62 1	5 *nm*	0 667	0 286	6 792
62 4	7 *nm*	0 416	1 272	6 542
62 56		0 468	3 057	

7 Numerical Example: Minimum Terminal Uncertainty in Translunar Navigation through Optimal Measurement Sequencing

In this section we present a numerical example of the problem in Sec 3 We seek the sequence of measurements that will 1) minimize the terminal mean-square uncertainty in position, or 2) minimize a weighted sum of the terminal mean square uncertainties in position and in velocity for a translunar mission We treat only the navigation problem, assuming the trajectory to be entirely free-fall Only one function of the state is measured: the angle between the line-of-sight to a star and the far or near horizon of the earth or moon This measurement, differing from the presentation of Sec 3, is made at a discrete set of (prescribed) times along the trajectory, with only a discrete set of stars which may be sighted $\Theta(t)$ is a vector of parameters which have only a discrete collection of possible values at a discrete set of times

The differential equation (3 7) is no longer appropriate The difference equation that governs the development of E is

$$E_n = E_n' - E_n'H_n(H_n^TE_n'H_n + R_n)^{-1}H_n^TE_n' \quad (7\ 1)$$

where the subscript n refers to the sampling time t_n, n running from 1–N Considering only the navigation problem, we have taken $Q(t)$ to be zero The quantity E_n' (following Battin[2]) is

$$E_n' = \Phi_{n\ n-1}E_{n-1}\Phi_{n\ n-1}^T \quad (7\ 2)$$

where $\Phi_{n\ n-1}$ is the transition matrix $\Phi(t_n,t_{n-1})$ This is determined from the preplanned or nominal trajectory If the initial condition $E(0)$ and measurements Θ_n with H_n and R_n are given, the entire E_n history is determined The mean-square uncertainty in position is

$$\mathrm{tr}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}E$$

where I is the 3×3 identity matrix The weighted sum of position and velocity uncertainties is written

$$\mathrm{tr}\begin{pmatrix} I & 0 \\ 0 & cI \end{pmatrix}E$$

where c determines the relative weighting Both cases may be written as $\mathrm{tr}LE$ where the subscript E on L has been omitted for notational simplicity The problem is to choose Θ_n so as to minimize $\mathrm{tr}L_NE_N$ for appropriate L_N

To establish the response function for changes in the measurement sequence, we first write the difference equation for linearized perturbations of E_n:

$$\delta E_n = \delta E_n' - \delta E_n' H_n a_n^{-1} H_n^T E_n' - E_n' \delta H_n a_n^{-1} H_n^T E_n' + E_n' H_n a_n^{-1} (\delta H_n^T E_n' H_n + H_n^T \delta E_n' H_n + H_n^T E_n' \delta H_n + \delta R_n) a_n^{-1} H_n^T E_n' - E_n' H_n a_n^{-1} \delta H_n^T E_n' - E_n' H_n a_n^{-1} H_n^T \delta E' \quad (7\ 3)$$

where

$$a_n = H_n^T E_n' H_n + R_n \quad (7\ 4)$$

We seek the expression for

$$\mathrm{tr} L_n \delta E_n - \mathrm{tr} L_{n-1} \delta E_{n-1}$$

such that $\mathrm{tr} L_N \delta E_N$ is written as a sum of δH_n and δR_n terms Premultiplying (7 3) by L_n, and taking the trace, we obtain

$$\mathrm{tr} L_n \delta E_n = \mathrm{tr}[\Phi_{n\ n-1}^T (L_n - A_n L_n - L_n A^T + A_n L_n A_n^T) \Phi_{n\ n-1} \delta E_{n-1}] - 2\ \mathrm{tr} B_n L_n E_n \delta H_n + \mathrm{tr} B_n L_n B_n^T \delta R_n \quad (7\ 5)$$

where extensive use has been made of the invariance of the trace under cyclic permutation of the factors and where

$$A_n = H_n a^{-1} H_n^T E_n'$$

$$B_n = a_n^{-1} H_n^T E_n'$$

We choose the adjoint matrix difference equation as

$$L_{n-1} = \Phi_{n\ n-1}^T [L_n - A_n L_n - L_n A_n^T + A_n L_n A_n^T] \Phi_{n\ n-1} \quad (7\ 6)$$

Postmultiplying (7 6) by δE_{n-1} and taking the trace, we obtain

$$\mathrm{tr} L_{n-1} \delta E_{n-1} = \mathrm{tr} \Phi_{n\ n-1}^T [L_n - A_n L - L_n A_n^T + A_n L_n A_n^T] \Phi_{n\ n-1} \delta E_{n-1} \quad (7\ 7)$$

Subtracting (7 7) from (7 5), we obtain

$$\mathrm{tr} L_n \delta E_n - \mathrm{tr} L_{n-1} \delta E_{n-1} = -2\ \mathrm{tr} B_n L_n E_n \delta H_n + \mathrm{tr} B_n L_n B_n^T \delta R_n \quad (7\ 8)$$

Summing this difference equation, starting from t_N, we obtain

$$\mathrm{tr} L_N \delta E_N - \mathrm{tr} L_0 \delta E_0 = \sum_{n=1}^{N-1} (\mathrm{tr} \lambda_n^{(1)} \delta H_n + tr \lambda_n^{(2)} \delta R_n) \quad (7\ 9)$$

where

$$\lambda_n^{(1)} = -2 B_n L_n E_n \qquad \lambda_n^{(2)} = B_n L_n B_n^T$$

Both H_n and R_n are determined from the choice of measurement that is made Identifying the measurements by use of the parameter Θ_n, we write

$$\mathrm{tr} L_N \delta E_N = \sum_{n=1}^{N-1} \left[\frac{\partial}{\partial \Theta} [\mathrm{tr}(\lambda_n^{(1)} H_n + \lambda_n^{(2)} R\ )] \right] \delta \Theta_n \quad (7\ 10)$$

where we assume E_0 is given, so that $\delta E_0 = 0$ We do not expect to make the coefficient of $\delta\Theta_n$ in (7 10) equal to zero The optimal Θ_n sequence is achieved when no $\delta\Theta_n$ can improve the performance index

The computational procedure for obtaining the optimal sequence of measurements for this translunar navigation problem begins with the calculation of E_n from (7 1), using a nominal Θ_n sequence With H_n, R_n, and E_n the adjoint matrix L_n is calculated, starting at t_N where L_N is chosen to make $\mathrm{tr} L_N E_N$ the desired performance index $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ are determined simultaneously with L_n The sum $\lambda_n^{(1)} \delta H_n + \lambda_n^{(2)} \delta R_n$ is obtained for every change in Θ_n that satisfies

$$\delta \alpha_n^T W_n \delta \alpha_n \leq (dP_n)^2 \quad (7\ 11)$$

where

$$\delta \alpha_n = \begin{pmatrix} \delta H_n \\ \delta R_n \end{pmatrix}$$

and dP_n is chosen empirically so that the changes δH_n, δR_n will not cause the linearization of the perturbation equations to be invalid At each t_n the change in measurement $\delta\Theta_n$ is chosen from among those which satisfy (7 11) to minimize $\lambda_n^{(1)} \delta H_n + \lambda_n^{(2)} \delta R_n$, thus giving the maximum predicted reduction in $\mathrm{tr} L_N E_N$

For this problem Θ is a 3-vector The choice of star to be sighted is 1 parameter In this example there was a catalog of 20 stars available The choice of earth or moon horizon is the second parameter The choice of near or far horizon is the third Thus, the Θ vector has 80 possible values Not all of the stars are available at a given t_n, however, because they may be blocked from view or they may be too close to the sun In fact, there were never more than 30 available values of Θ_n The initial E matrix was selected as

$$E_0 = \begin{bmatrix} 0\ 918\ (\text{miles})^2, & 0\ 063\ (\text{miles})^2, & 0\ 203\ (\text{miles})^2, & 0 & 0 & 0 \\ 0\ 063\ (\text{miles})^2, & 4\ 58\ (\text{miles})^2, & -1\ 86\ (\text{miles})^2, & 0 & 0 & 0 \\ 0\ 203\ (\text{miles})^2, & -1\ 86\ (\text{miles})^2, & 7\ 04\ (\text{miles})^2, & 0 & 0 & 0 \\ 0 & 0 & 0 & (7\ 73\ \text{miles/hr})^2, & (4\ 65\ \text{miles/hr})^2, & (2\ 72\ \text{miles/hr})^2, \\ 0 & 0 & 0 & (4\ 65\ \text{miles/hr})^2, & (83\ 8\ \text{miles/hr})^2, & (36\ 0\ \text{miles/hr})^2, \\ 0 & 0 & 0 & (2\ 72\ \text{miles/hr})^2, & (36\ 0\ \text{miles/hr})^2, & (36\ 1\ \text{miles/hr})^2, \end{bmatrix} \quad (7\ 12)$$

As the first example, the performance index was taken as the terminal mean square uncertainty in position The measurement sequence determined by Battin's method[2] was used as the nominal The values of Θ_n used in the nominal measurement sequence, the nominal mean-square position and velocity uncertainties, and the nominal magnitude, in the Cartesian sense, of the vector $\lambda_n^{(1)}$ are given in Table 1 The same information for the optimized measurement program is given in Table 2 An 18 6% reduction is achieved in the terminal mean-square position uncertainty We note that on the nominal trajectory the first several measurements reduce the velocity uncertainty while the position uncertainty reaches larger values The last several measurements primarily reduce the position uncertainty The optimized measurement sequence of Table 2 behaves in the same manner Both the position and velocity uncertainties, however, are larger than those of Table 1 throughout most of the trajectory except, of course, at the end The magnitude of $\lambda_n^{(1)}$ is much larger on the last ten than on the earlier measurements, showing that the greatest sensitivity of terminal position uncertainty to the measurements used occurs at the end of the trajectory $\lambda_n^{(1)} \delta H_n + \lambda_n^{(2)} \delta R_n$ is not zero for the optimal measurement program, but there are no alternate measurements for which it is negative This indicates an optimal solution

In Table 1, note that the nominal sequence of measurements taken oscillates between sighting the earth and the moon horizons One previous view was that more information was obtained by this arrangement Table 2 contradicts this hypothesis by making a series of all earth horizon sightings and then all moon horizon sightings The change from earth sightings to moon sightings occurs at the same measurement time in Tables 2 and 3 and the twelfth iteration of Table 4

In minimizing the weighted sum of terminal mean-square position and velocity uncertainties, the weighting constant

Table 4 A sequence of measurement program iterations

Measurement time, hr	Nominal $\text{tr}L_NE_N = 3216.4$ miles²		3rd Iteration $\text{tr}L_NE_N = 0.58416$ miles²		9th Iteration $\text{tr} = L_NE_N = 0.47646$ miles²		12th Iteration $\text{tr}L_NE_N = 0.47616$ miles²	
	Parameter values defining the measurements	Response function $\lvert\lambda_n^{(1)}\rvert$, (miles³/rad) × 10²	Parameter values defining the measurements	Response function $\lvert\lambda_n^{(1)}\rvert$, (miles³/rad) × 10²	Parameter values defining the measurements	Response function $\lvert\lambda_n^{(1)}\rvert$, (miles³/rad) × 10²	Parameter values defining the measurements	Response function $\lvert\lambda_n^{(1)}\rvert$, (miles³/rad) × 10²
0								
0.6	1 *ne*	63,613	5 *ne*	0.663	10 *ne*	0 105	10 *ne*	0 083
0.9	1 *ne*	87,187	5 *ne*	0.700	10 *ne*	0.136	14 *ne*	0 113
1.2	1 *ne*	95,965	5 *ne*	0.545	10 *ne*	0.145	14 *ne*	0 121
1.5	1 *ne*	94,697	5 *ne*	0.301	1 *ne*	0.094	1 *ne*	0 099
1.8	1 *ne*	87,172	5 *ne*	0.041	10 *ne*	0.132	8 *ne*	0 114
2.2	1 *ne*	71,628	5 *ne*	0.319	5 *ne*	0.184	5 *ne*	0.114
2.6	1 *ne*	52,857	5 *ne*	0.621	5 *ne*	0.168	5 *ne*	0.102
3.0	1 *ne*	32,627	5 *ne*	0.871	1 *fe*	0.062	8 *fe*	0 085
3.4	1 *ne*	11,766	5 *ne*	1.070	1 *fe*	0.051	10 *fe*	0 066
3.8	1 *ne*	9,380	5 *fe*	1.054	7 *ne*	0.038	7 *ne*	0 054
4.5	1 *ne*	46,854	5 *fe*	1.260	7 *ne*	0.033	7 *ne*	0.043
5.5	1 *ne*	101,925	5 *fe*	1.395	7 *ne*	0.027	5 *fe*	0.037
6.0	1 *ne*	130,492	5 *fe*	1.413	7 *ne*	0.024	5 *fe*	0.030
6.5	1 *ne*	159,929	5 *fe*	1.406	7 *ne*	0.021	5 *fe*	0.024
7 0	1 *ne*	190,329	5 *fe*	1.380	7 *ne*	0.018	5 *fe*	0.018
7.5	1 *ne*	221,748	15 *fe*	1.136	7 *fe*	0.013	5 *ne*	0.014
8.5	1 *ne*	287,734	15 *fe*	0.926	7 *fe*	0.009	5 *ne*	0.008
9.5	1 *ne*	357,857	15 *fe*	0.750	14 *fe*	0.009	5 *ne*	0.005
10.0	1 *ne*	394,396	15 *fe*	0.672	7 *fe*	0.007	8 *fe*	0.018
10.5	1 *ne*	431,865	15 *fe*	0.602	15 *ne*	0.220	15 *ne*	0.023
12.0	1 *ne*	549,348	14 *ne*	0.610	15 *fe*	0.301	15 *fe*	0.033
12.5	1 *ne*	590,029	16 *mn*	0.313	15 *fe*	0.340	15 *fe*	0.038
13.5	1 *ne*	673,357	15 *ne*	0.279	15 *fe*	0.043	15 *fe*	0.048
15.0	1 *ne*	802,486	15 *ne*	0.189	15 *ne*	0.059	8 *ne*	0 068
16.0	1 *ne*	890,771	20 *ne*	0.177	20 *ne*	0.073	20 *ne*	0.081
17 0	1 *ne*	980,375	20 *ne*	0.136	20 *ne*	0.085	20 *ne*	0.095
19.5	1 *ne*	1,207,959	15 *fe*	0.088	8 *fe*	0.125	8 *fe*	0.128
22.0	1 *ne*	1,436,778	8 *fe*	0.091	8 *fe*	0.163	8 *fe*	0.166
23.5	1 *ne*	1,572,757	8 *fe*	0.111	8 *fe*	0.186	8 *fe*	0.189
28.5	1 *ne*	2,005,401	8 *fe*	0.320	16 *fm*	0.326	16 *fm*	0.357
29.5	1 *ne*	2,085,973	8 *fe*	0.389	16 *fm*	0.369	16 *fm*	0 401
37 0	1 *ne*	2,577,575	6 *fm*	1.454	16 *fm*	0.839	16 *fm*	0.886
40.5	1 *ne*	2,698,627	6 *nm*	2.608	16 *fm*	1.195	16 *fm*	1.248
53.5	1 *ne*	962,109	3 *nm*	13.718	16 *fm*	4.633	20 *fm*	4.696
57.5	1 *ne*	3,030,453	17 *fm*	20 464	16 *nm*	6.718	16 *nm*	6.771
60.0	1 *ne*	12,110,476	17 *nm*	25.027	20 *fm*	8.341	20 *fm*	8.349
60.5	1 *ne*	16,342,105	3 *nm*	19.221	6 *fm*	4.453	6 *fm*	4.449
61.4	1 *ne*	31,415,158	15 *nm*	9.514	8 *nm*	2.612	8 *nm*	2.623
62.1	1 *ne*	65,901,113	16 *fm*	6.885	5 *nm*	4.668	5 *nm*	4.693
62.4	2 *nm*	61,148	7 *nm*	5.044	7 *nm*	4.032	7 *nm*	4.024
62.56								

c was taken as 0 138 (hr)^2, making

$$L_N = \begin{pmatrix} I & 0 \\ 0 & 0\ 138(\text{hr})^2\ I \end{pmatrix} \quad (7\ 13)$$

Table 3 gives the same information for this case as in Table 1 The decrease from the nominal $\text{tr} L_N E_N$ was 29 1% The mean-square position uncertainty was reduced 16 4%; the mean-square velocity uncertainty was reduced 39 3%

A second nominal was chosen in which only one star and the near horizon of the earth is measured throughout the trajectory (except at the last measurement time***) The terminal mean-square position uncertainty was 3216 (miles)^2 and the terminal mean-square velocity uncertainty was 8572 (mph)^2 The program iterated 12 times before the minimum terminal mean-square position uncertainty was obtained Table 4 illustrates significant iterations by listing the stars used in the measurement, the magnitude of $\lambda_n^{(1)}$, $\text{tr}L_N E_N$, and the iteration number The convergence properties of this optimization scheme show up well in that the performance index was reduced from 3216–0 5842 (miles)^2 by the third iteration The magnitude of $\lambda_n^{(1)}$ also decreased quite rapidly as shown by Table 4

The $\text{tr}L_N E_N$ oscillated during the seventh, eighth, ninth, and tenth iterations, indicating that the measurement program had become close to optimal The ninth iteration is shown in Table 4 The values of the $\text{tr}L_N E_N$ for the seventh through tenth iterations were 0 4766, 0 4828, 0 4765, and 0 4884 (miles)^2, respectively The majority of measurements are the same for the seventh and ninth iterations and the eighth and tenth iterations With a smaller dP_n for the tenth iteration than was used, a small improvement in the ninth iteration could have been made, stopping the oscillation The program iterates until no value of $\delta\Theta_n$ with the dP_N limit can be found such that $\lambda_n^{(1)}\delta H_n + \lambda_n^{(2)}\delta R_n$ is negative This minimum value of the performance index obtained on the twelfth iteration is shown in Table 4 with $\text{tr}L_N E_N = 0\ 4762$ (miles)^2 Note that this minimum value of $\text{tr}L_N E_N$ is slightly larger than that of Table 2, but still better than the nominal of Table 1 obtained by Battin's method The minimization of the performance index is based upon the linearization of Eq (7 1), which assumes that only changes in δH which lead to first-order changes in $\text{tr}L_N E_N$ can be made This is not necessarily the case in the navigational guidance problem where only a discrete set of measurements is given Thus, $\lambda_n^{(1)}\delta H_n + \lambda_n^{(2)}\delta R_n$ can predict a degradation of the performance index for all possible measurement changes where a small improvement actually could be made

*** At the last measurement time, the star could not be observed and another star was arbitrarily chosen

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